Topology and its Applications 158 (2011) 1493–1502

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Dimensions $\mathcal{K}$-Ind and $\mathcal{L}$-Ind. Some answers

V.V. Fedorchuk [1]

Moscow State University, Russia

A R T I C L E I N F O

MSC:
54F45

Keywords:
Dimension
Simplicial complex
Polyhedron
Homotopy domination

A B S T R A C T

We solve some problems concerning dimension function $\mathfrak{K}$-Ind ($\mathfrak{K}$ is a class of finite simplicial complexes) and $\mathfrak{L}$-Ind ($\mathfrak{L}$ is a class of compact polyhedra). One of the main results is:

If $\mathfrak{L}_1$ is homotopy dominated by $\mathfrak{L}_2$, then $\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant \mathfrak{L}_2\text{-}\operatorname{Ind} X$ for every normal space X.

1. Introduction

In [9,10] new dimension inductive functions $\mathfrak{K}$-Ind and $\mathfrak{L}$-Ind, where $\mathfrak{K}$ is a non-empty set of finite simplicial complexes and $\mathfrak{L}$ is a non-empty set of compact polyhedra, were introduced (see Definitions 2.10 and 2.12). Their transfinite versions tr-$\mathfrak{K}$-Ind and tr-$\mathfrak{L}$-Ind were defined in [11] (see Definitions 2.14 and 2.16). If $\mathfrak{K} = \mathfrak{L} = \{S^0\}$, then

$$\mathfrak{K}\text{-}\operatorname{Ind} X = \mathfrak{L}\text{-}\operatorname{Ind} X = \operatorname{Ind} X; \tag{1.1}$$

$$tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X \tag{1.2}$$

for an arbitrary normal space X, where $tr\text{-}\operatorname{Ind} X$ is the large transfinite inductive dimension. For arbitrary sets $\mathfrak{K}$ and $\mathfrak{L}$ we have

$$\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant \operatorname{Ind} X, \qquad \mathfrak{L}\text{-}\operatorname{Ind} X \leqslant \operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant tr\text{-}\operatorname{Ind} X, \qquad tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant tr\text{-}\operatorname{Ind} X.$$

In [9] (respectively in [11]) it was proved that for every normal space X,

$$\left.\begin{array}{l} \mathfrak{K}\text{-}\operatorname{Ind} X = \operatorname{Ind} X \\ tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X \end{array}\right\} \Leftrightarrow \quad \mathfrak{K} \text{ contains a disconnected complex.} \tag{1.3}$$

As for functions $\mathfrak{L}$-Ind and tr-$\mathfrak{L}$-Ind, the corresponding statement was proved only for hereditarily normal spaces X. Here we prove (Theorems 3.9 and 3.10) that for every normal space X,

$$\left.\begin{array}{l} \mathfrak{L}\text{-}\operatorname{Ind} X = \operatorname{Ind} X \\ tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X \end{array}\right\} \Leftrightarrow \quad \mathfrak{L} \text{ contains a disconnected polyhedron.} \tag{1.4}$$

E-mail address: vvfedorchuk@gmail.com.

[1] The author was supported by the Russian Foundation for Basic Research (Grant 09-01-00741) and the Program "Development of the Scientific Potential of Higher School" of the Ministry for Education of the Russian Federation (Grant 2.1.1.3704).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.05.021

In [9] (respectively in [11]) it was proved that for every hereditarily normal X,

$$\mathfrak{L}_1 \leqslant_h \mathfrak{L}_2 \quad \Rightarrow \quad \mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant \mathfrak{L}_2\text{-}\operatorname{Ind} X; \tag{1.5}$$

$$\mathfrak{L}_1 \leqslant_h \mathfrak{L}_2 \quad \Rightarrow \quad tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X. \tag{1.6}$$

Here we prove inequalities (1.5) and (1.6) for an arbitrary normal space X (Theorems 3.4 and 3.2 respectively).

In [11] there were defined classes I-c and tr-I-c of simplicial complexes and I-p and tr-I-p of compact polyhedra (see Definitions 3.12 and 3.13). Questions about relationships between these classes were formulated. Theorems 3.15 and 3.16 give complete answers to these questions. Namely:

$$I\text{-}c = tr\text{-}I\text{-}c = \mathfrak{K}_d;$$

$$I\text{-}p = tr\text{-}I\text{-}p = \mathfrak{L}_d.$$

Here $\mathfrak{K}_d$ is the class of all disconnected simplicial complexes and $\mathfrak{L}_d$ is the class of all disconnected compact polyhedra. In Section 4 we continue the study of (tr)-I-complexes and (tr)-I-polyhedra, and ask some questions.

2. Preliminaries

2.1. All spaces are normal and T_1. All mappings are continuous. For a space X by $\exp X$ we denote the set of all closed subsets of X (including $\emptyset$). By $\operatorname{Fin}_s(\exp X)$ we denote the set of all finite sequences $\Phi = (F_1, \dots, F_m)$, $F_j \in \exp X$.

Complexes are finite complete simplicial complexes. Recall that a simplicial complex K is said to be *complete* if K contains all faces of every simplex belonging to K. For a complex K by $L(K)$ we denote the underlying polyhedron. In what follows *polyhedra* stand for compact polyhedra.

By $v(K)$ we denote the set of all vertices of a complex K. The *nerve* of a finite family u of sets is denoted by $N(u)$.

2.2. Definition. ([7]) Let X be a space, K be a complex, and $\Phi = (F_1, \dots, F_m) \in \operatorname{Fin}_s(\exp X)$. A sequence $u = (U_1, \dots, U_k)$, $k \geqslant m$, of open subsets of X is called a K-*neighbourhood* of Φ if $F_j \subset U_j$ and there is an embedding $N(u) \subset K$. One can number vertices $a_j \in v(K)$ so that the embedding $N(u) \subset K$ is defined by the correspondence $U_j \to a_j$.

2.3. Definition. ([7]) A set $P \subset X$ is said to be a K-*partition* of $\Phi \in \operatorname{Fin}_s(\exp X)$ (notation: $P \in \operatorname{Part}(\Phi, K)$) if $P = X \setminus \bigcup u$, where u is a K-neighbourhood of Φ.

Put

$$\operatorname{Exp}_K(X) = \{\Phi \in \operatorname{Fin}_s(\exp X):\ N(\Phi) \subset K\}. \tag{2.1}$$

2.4. Open swelling lemma. *If $\Phi = (F_1, \dots, F_m) \in \operatorname{Fin}_s(\exp X)$, then there exists a family $u = (U_1, \dots, U_m)$ of open subsets of X such that*

$$F_j \subset U_j, \quad j = 1, \dots, m;$$

$$N(\Phi) = N(u).$$

Lemma 2.4 yields

2.5. Proposition. *If $\Phi \in \operatorname{Exp}_K(X)$, then there exists a K neighbourhood u of Φ.*

2.6. Definition. Let F be a closed subset of a space X and let L be a polyhedron. A mapping $f: F \to L$ is called a *partial mapping* of X to L (notation: $f \in PM(X, L)$).

2.7. Every polyhedron L is an *ANR*-compactum. Consequently, $L \in ANE$ for normal spaces. It means that any $f \in PM(X, L)$ can be extended over some open set $U \supset \operatorname{dom} f$.

2.8. Definition. Let $f \in PM(X, L)$. A closed set $P \subset X$ is called a *partition* of f (notation: $P \in \operatorname{Part}(f, L)$), if $P \cap \operatorname{dom} f = \emptyset$ and f can be extended over $X \setminus P$.

From 2.7 it follows that every $f \in PM(X, L)$ has a partition.

2.9. Definition. Let $\mathcal{L}$ be a certain set of polyhedra, $L_i \in \mathfrak{L}$, $f_i \in PM(X, L_i)$, $i = 1, \dots, r$. The set $\{f_1, \dots, f_r\}$ is called $\mathfrak{L}$-*inessential*, if there exist partitions $P_i \in \operatorname{Part}(f_i, L_i)$ such that $P_1 \cap \dots \cap P_r = \emptyset$.

2.10. Definition. Let $\mathcal{L}$ be a non-empty set of polyhedra. To every space X one assigns the dimension $\mathfrak{L}\text{-}\dim X$ which is an integer $\geqslant -1$ or ∞. The dimension function $\mathfrak{L}\text{-}\dim$ is defined in the following way:

(1) $\mathfrak{L}\text{-}\dim X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathfrak{L}\text{-}\dim X \leqslant n$, where $n = 0, 1, \ldots,$ if every set $\{f_i \in PM(X, L_i)\colon\ L_i \in \mathcal{L},\ i = 1, \ldots, n+1\}$ is $\mathfrak{L}$-inessential;
(3) $\mathfrak{L}\text{-}\dim X = \infty$, if $\mathfrak{L}\text{-}\dim X > n$ for all $n \geqslant -1$.

If the set $\mathfrak{L}$ contains only one compactum L we write $\mathfrak{L} = L$ and $\mathfrak{L}\text{-}\dim X = L\text{-}\dim X$.

From characterization of the Lebesgue dimension by means of partitions we get

2.11. Theorem. *For every space X, $S^0\text{-}\dim X = \dim X$.*

2.12. Definition. ([11]) Let $\mathfrak{K}$ be a non-empty set of complexes. To every space X one assigns the dimension $\mathfrak{K}\text{-}\operatorname{Ind} X$ which is an integer $\geqslant -1$ or ∞. The dimension function $\mathfrak{K}\text{-}\operatorname{Ind}$ is defined in the following way:

(1) $\mathfrak{K}\text{-}\operatorname{Ind} X = -1 \Leftrightarrow X = \emptyset$;
(2) $\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant n$, where $n = 0, 1, \ldots,$ if for every $K \in \mathfrak{K}$ and $\Phi \in \operatorname{Exp}_K(X)$ there exists a partition $P \in \operatorname{Part}(\Phi, K)$ such that $\mathfrak{K}\text{-}\operatorname{Ind} P \leqslant n-1$;
(3) $\mathfrak{K}\text{-}\operatorname{Ind} X = \infty$, if $\mathfrak{K}\text{-}\operatorname{Ind} X > n$ for all $n \geqslant -1$.

If the set $\mathfrak{K}$ contains only one complex K we write $\mathfrak{K} = K$ and $\mathfrak{K}\text{-}\operatorname{Ind} X = K\text{-}\operatorname{Ind} X$.

Let $\{0, 1\}$ be a simplicial complex consisting of two points. From definition of the large inductive dimension Ind we get

2.13. Theorem. *For every space X, $\{0, 1\}\text{-}\operatorname{Ind} X = \operatorname{Ind} X$.*

By Ord we denote the class of all ordinal numbers.

2.14. Definition. Let $\mathfrak{K}$ be a non-empty set of complexes. To every space X one assigns the transfinite dimension $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X$ which is -1, ordinal number α, or ∞. The dimension function $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind}$ is defined in the following way:

(1) $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = -1 \Leftrightarrow X = \emptyset$;
(2) $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant \alpha$, $\alpha \in \mathrm{Ord}$, if for every $K \in \mathfrak{K}$ and $\Phi \in \operatorname{Exp}_K(X)$ there exists a partition $P \in \operatorname{Part}(\Phi, K)$ such that $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} P < \alpha$;
(3) $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = \infty$, if $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X > \alpha$ for all $\alpha \in \mathrm{Ord}$.

If $\mathfrak{K}$ contains only one complex K we write $\mathfrak{K} = K$ and $tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = tr\text{-}K\text{-}\operatorname{Ind} X$.

2.15. Theorem. *For every space X, $tr\text{-}\{0, 1\}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X$.*

2.16. Definition. Let $\mathfrak{L}$ be a non-empty set of polyhedra. If we substitute in Definition 2.14 partitions $P \in \operatorname{Part}(\Phi, K)$ by partitions $P \in \operatorname{Part}(f, L)$, we obtain definition of function $tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind}$.

2.17. Theorem. *For every space X,*

$$S^0\text{-}\operatorname{Ind} X = \operatorname{Ind} X;$$

$$tr\text{-}S^0\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X.$$

2.18. Theorem. *([9]) For every space X,*

$$\mathfrak{L}\text{-}\dim X \leqslant \mathfrak{L}\text{-}\operatorname{Ind} X;$$

$$\mathfrak{L}\text{-}\dim X \leqslant 0 \quad \Leftrightarrow \quad \mathfrak{L}\text{-}\operatorname{Ind} X \leqslant 0.$$

2.19. Theorem. *([9–11]) For every space X,*

$$\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant \operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant tr\text{-}\operatorname{Ind} X;$$

$$\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant \operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X \leqslant tr\text{-}\operatorname{Ind} X.$$

2.20. Theorem. *([9–11]) If $\mathfrak{L}_1 \subset \mathfrak{L}_2$, then*

$$\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant \mathfrak{L}_2\text{-}\operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X$$

for every space X.

2.21. Theorem. *([9–11]) Each of the following equalities*

$$\mathfrak{K}\text{-}\operatorname{Ind} X = \operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{K}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X$$

hold for every space X if and only if $\mathfrak{K}$ contains a disconnected complex K.

Let $\mathfrak{L}$ be a non-empty set of polyhedra. For each $L \in \mathfrak{L}$ we fix a triangulation $t = t(L)$ of $\mathfrak{L}$. The pair (L, t) is a simplicial complex which is denoted by L_t. The family $\tau = \{t(L)\colon L \in \mathfrak{L}\}$ is said to be a *triangulation* of the set $\mathfrak{L}$. Let $\mathfrak{L}_\tau = \{L_t\colon t \in \tau\}$.

2.22. Theorem. *([9–11]) Let $\mathfrak{L}$ be a non-empty set of polyhedra and let τ be some its triangulation. Then*

$$\mathfrak{L}_\tau\text{-}\operatorname{Ind} X \leqslant \mathfrak{L}\text{-}\operatorname{Ind} X;$$

$$trV\mathfrak{L}_\tau\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X$$

for every space X.

If X is a hereditarily normal space, then

$$\mathfrak{L}_\tau\text{-}\operatorname{Ind} X = \mathfrak{L}\text{-}\operatorname{Ind} X;$$

$$tr\text{-}\mathfrak{L}_\tau\text{-}\operatorname{Ind} X = tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X.$$

Moreover, for every space X,

$$\mathfrak{L}_\tau\text{-}\operatorname{Ind} X \leqslant 0 \quad \Leftrightarrow \quad \mathfrak{L}\text{-}\operatorname{Ind} X \leqslant 0.$$

2.23. The first inverse system theorem. *([8]) If X is the limit space of an inverse system $\{X_\alpha, \pi^a_b, A\}$ of compact spaces X_α such that $\mathfrak{L}\text{-}\dim X_\alpha \leqslant n$ for all $a \in A$, then $\mathfrak{L}\text{-}\dim X \leqslant n$.*

2.24. The second inverse system theorem. *([8]) Let X be a compact space and let $\mathfrak{L}$ be a set of polyhedra such that $\mathfrak{L}\text{-}\dim X \leqslant n$. Then X is the limit space of an inverse system $\{X_\alpha, \pi^a_b, A\}$ consisting of metrizable compacta X_α such that $\mathfrak{L}\text{-}\dim X_\alpha \leqslant n$ for every $\alpha \in A$.*

2.25. Definition. ([12]) Let B^{n+1} be a closed ball and let S^n be its boundary. A space X is said to be C^n-*space* (notation: $X \in C^n$) if every mapping $f : S^n \to X$ can be extended over X.

A space X is called LC^n-space if for every point $x \in X$ and neighbourhood Ox there exists a neighbourhood $O_1x \subset Ox$ such that for each mapping $f : S^n \to O_1x$ there exists a mapping $\bar{f} : B^{n+1} \to Ox$ with $\bar{f}|_{S^n} = f$.

2.26. Definition. A space L is an *absolute extensor* for a space X (notation: $L \in AE(X)$) if every $f \in PM(X, L)$ can be extended over X.

2.27. Theorem. *([12]) If $L \in C^n \cap LC^n$, then $L \in AE(X)$ for every metrizable compactum X with* $\dim X \leqslant n + 1$.

In particular, if L is a connected polyhedron, then $L \in AE(X)$ for every metrizable compactum X with $\dim X \leqslant 1$.

2.28. Borsuk's theorem on extension of homotopy. *([13,14]) If F is a closed subset of a space X, then each mapping*

$$f : (X \times 0) \cup (F \times I) \to L$$

into ANR-compactum L extends over $X \times I$.

2.29. Theorem. *([8]) For every space X,*

$$\mathfrak{L}\text{-}\dim X = \mathfrak{L}\text{-}\dim \beta X.$$

3. Main results

3.1. Definition. We say that a set $\mathfrak{L}_1$ *is* (*homotopically*) *dominated* by a set $\mathfrak{L}_2$ (notation: $\mathfrak{L}_1 \leqslant_h \mathfrak{L}_2$) if every $L_1 \in \mathfrak{L}_1$ is dominated by some $L_2 \in \mathfrak{L}_2$. A set $\mathfrak{L}_1$ is homotopically *equivalent* to a set $\mathfrak{L}_2$ (notation: $\mathfrak{L}_1 \simeq \mathfrak{L}_2$) if both $\mathfrak{L}_1 \leqslant_h \mathfrak{L}_2$ and $\mathfrak{L}_2 \leqslant_h \mathfrak{L}_1$ hold.

3.2. Theorem. *If $\mathfrak{L}_1$ is homotopically dominated by $\mathfrak{L}_2$, then*

$$tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X$$

for every space X.

Proof. We apply induction on $tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X = \alpha \geqslant -1$. For $\alpha = -1$ the assertion is obvious. Let $tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X = \alpha \geqslant 0$. Take arbitrary $L_1 \in \mathfrak{L}_1$ and $f \in PC(X, L_1)$:

$$f : f \to L_1, \quad \text{where } F \text{ is a closed subset of } X.$$

We have to find a partition $P \in \operatorname{Part}(f, L_1)$ with

$$tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} P < \alpha.$$

By definition, L_1 is homotopically dominated by some $L_2 \in \mathfrak{L}_2$. There exist mapping $\alpha : L_1 \to L_2$ and $\beta : L_2 \to L_1$ such that

$$\beta \circ \alpha \text{ is homotopically equivalent to } \mathrm{id}_{L_1}. \tag{3.1}$$

Put

$$g = \alpha \circ f : F \to L_2. \tag{3.2}$$

Then $g \in PC(X, L_2)$. Since $tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X = \alpha$, there exists a partition $P \in \operatorname{Part}(g, L_2)$ with $tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} P < \alpha$. Consequently, there is a mapping $\bar{g} : X \setminus P \to L_2$ such that

$$\bar{g}|_F = g. \tag{3.3}$$

Consider a mapping $h = \beta \circ \bar{g} : X \setminus P \to L_1$.

We have $h|_F = \beta \circ \bar{g}|_F = (\text{in view of } (3.3)) = \beta \circ g = (\text{according to } (3.2)) = \beta \circ d \circ f$, i.e.

$$h|_F = (\beta \circ \alpha) \circ f. \tag{3.4}$$

From (3.1) it follows that

$$h|_F \text{ is homotopically equivalent to } f. \tag{3.5}$$

Let OF be a neighbourhood of F such that $\mathrm{Cl}(OF) \cap P = \emptyset$. Put $F_1 = F \cup \mathrm{Bd}(OF)$ and define a mapping $f_1 : F_1 \to L_1$ as follows

$$f_1|_F = f, \qquad f_1|_{\mathrm{Bd}(OF)} = h|_{\mathrm{Bd}(OF)}. \tag{3.6}$$

From (3.5) it follows that

$$h|_{F_1} \text{ is homotopically equivalent to } f_1.$$

So the mapping f_1 can be extended over $\mathrm{Cl}(OF)$ by Borsuk's homotopy extension theorem (Theorem 2.28). Denote this extension by $\bar{f}_1$. Now we denote a mapping $\bar{f} : X \setminus P \to L_1$ as follows

$$\bar{f}|_{\mathrm{Cl}(OF)} = \bar{f}_1, \qquad \bar{f}|_{X \setminus P \cup OF} = h|_{X \setminus P \cup OF}.$$

Since $\bar{f}_1|_{\mathrm{Bd}(OF)} = f_1|_{\mathrm{Bd}(OF)} = (\text{according to } (3.6)) = h|_{\mathrm{Bd}(OF)}$, the mapping $\bar{f}$ is correctly defined and continuous. Thus $P \in \operatorname{Part}(f, L_1)$. We have $tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} P \leqslant (\text{by inductive assumption}) \leqslant tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} P < \alpha$. $\square$

Theorem 3.2 yields

3.3. Theorem. *If $\mathfrak{L}_1 \simeq \mathfrak{L}_2$, then*

$$tr\text{-}\mathfrak{L}_1\text{-}\operatorname{Ind} X = tr\text{-}\mathfrak{L}_2\text{-}\operatorname{Ind} X.$$

The next two theorems are corollaries of Theorems 3.2 and 3.3 respectively.

3.4. Theorem. *If $\mathfrak{L}_1 \leqslant_h \mathfrak{L}_2$, then*

$$\mathfrak{L}_1\text{-}\operatorname{Ind} X \leqslant \mathfrak{L}_2\text{-}\operatorname{Ind} X.$$

3.5. Theorem. *If $\mathfrak{L}_1 \simeq \mathfrak{L}_2$, then*

$$\mathfrak{L}_1\text{-}\operatorname{Ind} X = \mathfrak{L}_2\text{-}\operatorname{Ind} X.$$

3.6. Remark. Theorems 3.2–3.5 answer Question 3.17 from [9] and Question 3.3 from [11]. Theorem 3.5 answers as well Question 8.3 from [10].

Theorem 3.2 implies also

3.7. Proposition. *If L_1 is a retract of L_2, then*

$$tr\text{-}L_1\text{-}\operatorname{Ind} X \leqslant tr\text{-}L_2\text{-}\operatorname{Ind} X;$$

$$L_1\text{-}\operatorname{Ind} X \leqslant L_2\text{-}\operatorname{Ind} X$$

for every space X.

3.8. Proposition. *If $\mathfrak{L}$-consists of connected polyhedra, then $\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant 0$ for every space X with* $\dim X \leqslant 1$.

Proof. Since $\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant 0$ if and only if $\mathfrak{L}\text{-}\dim X \leqslant 0$ (Theorem 2.18), it suffices to show that $\mathfrak{L}\text{-}\dim X \leqslant 0$ for every X with $\dim X \leqslant 1$.

If X is a metrizable compactum, then $L \in AE(X)$ for every $L \in \mathfrak{L}$ by Kuratowski theorem (Theorem 2.27). Hence

$$\mathfrak{L}\text{-}\dim X \leqslant 0 \quad \text{for each metric compactum } X \text{ with } \dim X \leqslant 1. \tag{3.7}$$

Now let X be an arbitrary one-dimensional space. Then $\dim \beta X \leqslant 1$. By Theorem 2.24 there exists an inverse system $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ consisting of metrizable compacta such that $\beta X = \lim S$ and $\dim X_\alpha \leqslant 1$ for every $\alpha \in A$. According to (3.7), $\mathfrak{L}\text{-}\dim X_\alpha \leqslant 0$ for any $\alpha \in A$. Consequently, $\mathfrak{L}\text{-}\dim \beta X \leqslant 0$ by the first inverse system theorem (Theorem 2.23). Then $\mathfrak{L}\text{-}\dim X \leqslant 0$ in view of Theorem 2.29. □

3.9. Theorem. *The equality $tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X$ holds for every space X if and only if $\mathfrak{L}$ contains a disconnected polyhedron.*

Proof. Necessity is a consequence of Proposition 3.8. Now let $\mathfrak{L}$ contain a disconnected polyhedron L. In accordance with Theorem 2.19, $tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X \leqslant tr\text{-}\operatorname{Ind} X$. So we have to check that $tr\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X$.

In view of Theorem 2.20, $tr\text{-}L\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X$. Since L is disconnected, there is a retraction $r : L \to S^0$. Proposition 3.7 implies that $tr\text{-}S^0\text{-}\operatorname{Ind} X \leqslant tr\text{-}L\text{-}\operatorname{Ind} X$. But $tr\text{-}S^0\text{-}\operatorname{Ind} X = tr\text{-}\operatorname{Ind} X$. So $tr\text{-}\operatorname{Ind} X \leqslant tr\text{-}L\text{-}\operatorname{Ind} X \leqslant tr\text{-}\mathfrak{L}\text{-}\operatorname{Ind} X$. □

Proposition 3.8 and Theorem 3.9 yield

3.10. Theorem. *The equality $\mathfrak{L}\text{-}\operatorname{Ind} X = \operatorname{Ind} X$ holds for every space X if and only if $\mathfrak{L}$ contains a disconnected polyhedron.*

3.11. Remark. Theorems 3.9 and 3.10 give positive answers to Question 3.16 from [9] and Question 3.4 from [11].

3.12. Definition. ([11]) A complex K is said to be an *I-complex* (*tr-I-complex*) (notation: $K \in I\text{-}c$ ($K \in tr\text{-}I\text{-}c$)) if for any space X we have

$$K\text{-}\operatorname{Ind} X < \infty \quad \Rightarrow \quad \operatorname{Ind} X < \infty$$

$$(tr\text{-}K\text{-}\operatorname{Ind} X < \infty \Rightarrow tr\text{-}\operatorname{Ind} X < \infty).$$

3.13. Definition. ([11]) A polyhedron L is called an *I-polyhedron* (*tr-I-polyhedron*) (notation: $L \in I\text{-}p$ $(L \in tr\text{-}I\text{-}p)$) if for any space X we have

$$L\text{-}\operatorname{Ind} X < \infty \quad \Rightarrow \quad \operatorname{Ind} X < \infty$$
$$(tr\text{-}L\text{-}\operatorname{Ind} X < \infty \ \Rightarrow \ tr\text{-}\operatorname{Ind} X < \infty).$$

3.14. Proposition. *There exist a locally compact space Y and a compact space Z such that*

$$\dim Y = \dim Z = 1;$$
$$tr\text{-}\operatorname{Ind} Y = \infty, \qquad \operatorname{Ind} Z = \infty.$$

Proof. Compact spaces X_n, $n = 2, 3, \ldots,$ were constructed in [15] such that

$$\dim X_n = 1 < n = \operatorname{Ind} X_n.$$

Let Y be equal to the discrete union $\bigoplus\{X_n\colon\ n = 2, 3, \ldots\}$ of the spaces X_n. Then $tr\text{-}\operatorname{Ind} Y = \infty$ according to the following

Statement. (See [3], for example.) If X is a discrete union of spaces Y_n, $n = 1, 2, \ldots,$ with $\operatorname{Ind} Y_n \geqslant n$, then $tr\text{-}\operatorname{Ind}\text{-}X$ is not defined.

To complete the proof, choose Z to be the Alexandroff compactification αY of the space Y. $\square$

Let

$$\mathfrak{K}_d = \{K\colon\ K \text{ is a disconnected complex}\};$$
$$\mathfrak{L}_d = \{L\colon\ L \text{ is a disconnected polyhedron}\}.$$

3.15. Theorem. *$I\text{-}c = tr\text{-}I\text{-}c = \mathfrak{K}_d$.*

Proof. The inclusions $\mathfrak{K}_d \subset I\text{-}c$, $\mathfrak{K}_d \subset tr\text{-}i\text{-}c$ are consequences of Theorem 2.21. Now let K be a connected complex and let $L(K)$ be the corresponding polyhedron. Then for the space Z from Proposition 3.14 we get

$K\text{-}\operatorname{Ind} Z \leqslant \text{(Theorem 2.22)} \leqslant L(K)\text{-}\operatorname{Ind} Z \leqslant \text{(Proposition 3.7)} \leqslant 0$. On the other hand, $\operatorname{Ind} Z = \infty$ by Proposition 3.14. Hence $I\text{-}c \subset \mathfrak{K}_d$.

We can prove the inclusion $tr\text{-}I\text{-}c \subset \mathfrak{K}_d$ in the same way considering the space Y instead of Z. $\square$

3.16. Theorem. *$I\text{-}p = tr\text{-}I\text{-}p = \mathfrak{L}_d$.*

The proof repeats the argument from the proof of Theorem 3.15.

3.17. Remark. Theorems 3.15 and 3.16 answer Questions 3.9–3.13 from [11].

4. Concluding remarks and questions

In this section we continue the study of (tr)-I complexes and (tr)-I-polyhedra.

4.1. Definition. Let $\mathfrak{X}$ be a class of spaces. A complex K is said to be an *$I(\mathfrak{X})$-complex* (*tr-$I(\mathfrak{X})$-complex*) (notation: $K \in I(\mathfrak{X})\text{-}c$ $(K \in tr\text{-}I(\mathfrak{X})\text{-}c)$) if for any $X \in \mathfrak{X}$ we have

$$K\text{-}\operatorname{Ind} X < \infty \quad \Rightarrow \quad \operatorname{Ind} X < \infty$$
$$(tr\text{-}K\text{-}\operatorname{Ind} X < \infty \ \Rightarrow \ tr\text{-}\operatorname{Ind} X < \infty).$$

A polyhedron L is called an *$I(\mathfrak{X})$-polyhedron* (*tr-$I(\mathfrak{X})$-polyhedron*) (notation: $L \in I(\mathfrak{X})\text{-}p$ $(L \in tr\text{-}I(\mathfrak{X})\text{-}p)$) if for any space $X \in \mathfrak{X}$ we have

$$L\text{-}\operatorname{Ind} X < \infty \quad \Rightarrow \quad \operatorname{Ind} X < \infty$$
$$(tr\text{-}L\text{-}\operatorname{Ind} X < \infty \ \Rightarrow \ tr\text{-}\operatorname{Ind} X < \infty).$$

4.2. Proposition. *If* $\mathfrak{X}_1 \subset \mathfrak{X}_2$, *then*

$$I(\mathfrak{X}_2)\text{-}c \subset I(\mathfrak{X}_1)\text{-}c;$$
$$tr\text{-}I(\mathfrak{X}_2)\text{-}c \subset tr\text{-}I(\mathfrak{X}_1)\text{-}c;$$
$$I(\mathfrak{X}_2)\text{-}p \subset I(\mathfrak{X}_1)\text{-}p;$$
$$tr\text{-}I(\mathfrak{X}_2)\text{-}p \subset tr\text{-}I(\mathfrak{X}_1)\text{-}p.$$

Let *Comp*, *hn Comp*, *pn Comp* be respectively the classes of all compact spaces, hereditarily normal compact spaces, perfectly normal compact spaces.

Proposition 4.2 yields

$$I\text{-}c \subset I(Comp)\text{-}c \subset I(hn\,Comp)\text{-}c \subset I(pn(Comp)\text{-}c; \tag{4.1}$$
$$tr\text{-}I\text{-}c \subset tr\text{-}I(Comp)\text{-}c \subset tr\text{-}I(hn\,Comp)\text{-}c \subset tr\text{-}I(pn\,Comp)\text{-}c; \tag{4.2}$$
$$I\text{-}p \subset I(Comp)\text{-}p \subset I(hn\,Comp)\text{-}p \subset I(pn\,Comp)\text{-}p; \tag{4.3}$$
$$tr\text{-}I\text{-}p \subset tr\text{-}I(Comp)\text{-}p \subset tr\text{-}I(hn\,Comp)\text{-}p \subset tr\text{-}I(pn\,Comp)\text{-}p. \tag{4.4}$$

From proofs of Theorems 3.15 and 3.16 it follows that

$$I\text{-}c = I(Comp)\text{-}c; \tag{4.5}$$
$$I\text{-}p = I(Comp)\text{-}p. \tag{4.6}$$

V. Chatyrko constructed [1,2] a snakelike continuum X such that $\dim X = 1$ (X is a snakelike continuum) and $tr\text{-}\operatorname{ind} X = \infty$. It implies that

$$tr\text{-}I\text{-}c = tr\text{-}I(Comp)\text{-}c; \tag{4.7}$$
$$tr\text{-}I\text{-}p = tr\text{-}I(Comp)\text{-}p. \tag{4.8}$$

4.3. Theorem. *Under* **CH** *we have*

$$I\text{-}c = I(pn\,Comp)\text{-}c; \tag{4.9}$$
$$tr\text{-}I\text{-}c = tr\text{-}I(pn\,Comp)\text{-}c; \tag{4.10}$$
$$I\text{-}p = I(pn\,Comp)\text{-}p; \tag{4.11}$$
$$tr\text{-}I\text{-}p = tr\text{-}I(pn\,Comp)\text{-}p. \tag{4.12}$$

To prove Theorem 4.3 it suffices to prove

4.4. Theorem (CH). *There exists a perfectly normal compact space* X *such that* $\dim X = 1$, $tr\text{-}\operatorname{Ind} X = \infty$.

To prove Theorem 4.4 we need an additional information. Let us recall some definitions.

4.5. Definition. ([4]) A mapping $f : X \to Y$ is called *fully closed* if the set

$f(F_1) \cap f(F_2)$ is discrete

for any closed sets $F_1, F_2 \subset X$ such that $F_1 \cap F_2 = \emptyset$.

It should be noted that Definition 4.5 is equivalent to the original definition of fully closed mappings from [4].

4.6. Definition. ([5]) A mapping $f : X \to Y$ is called *ring-like* if for any point $x \in X$ and arbitrary neighbourhoods Ox and Ofx, the set $f^{\#}Ox$ contains a partition between the point fx and the set $Y \setminus Ofx$ in the space Y. Here $f^{\#}Ox$ denotes the *small image* of Ox, that is, $f^{\#}Ox = Y \setminus f(X \setminus Ox)$.

4.7. Lemma. *([5]) Let* $f : X \to Y$ *be a ring-like mapping between compact spaces and let* C *be a non-degenerate subcontinuum of* Y*. If* B *is a closed subset of* X *such that* $f(B) = C$*, then* $B = f^{-1}C$.

4.8. Definition. ([6]) A closed mapping $f : X \to Y$ is called *full* if, for each closed set $C \subset Y$, the set $f(C) \setminus f^{\#}C$ is countable.

A part of the proof of Lemma 4 from [6] give us

4.9. Lemma. *Let $f : Z \to Y$ be a full, ring-like, fully closed mapping of a perfectly normal compact space Z onto a connected infinite Y. Then each partition P in Z contains the inverse image of some point $y \in Y$.*

4.10. Lemma. *Let $f : Z \to Y$ be a full, ring-like, fully closed mapping. Then, for any closed subset $Y_0 \subset Y$, the mapping*

$$f|_{f^{-1}Y_0} : f^{-1}Y_0 \to Y_0$$

is full, ring-like, and fully closed.

4.11. An inverse system $S = \{X_n,\ \pi_n^{n+1},\ n = 1, 2 \ldots\}$ of compact spaces was constructed in [6, Theorem 1] under **CH** with the following properties:

$$X_1 = I; \tag{4.13}$$

$$X_n \text{ is perfectly normal;} \tag{4.14}$$

$$\pi_n^{n+1} \text{ is fully closed, ring-like, and full;} \tag{4.15}$$

$$\left(\pi_n^{n+1}\right)^{-1} x \text{ is homeomorphic to } I \text{ for each point } x \in X_n; \tag{4.16}$$

$$\dim X_n = 1. \tag{4.17}$$

Proof of Theorem 4.4. The space X is the limit of the inverse system S from 4.11. This space is perfectly normal as the limit of an inverse sequence of perfectly normal compact space. From (4.17) it follows that $\dim X \leqslant 1$. All spaces X_n are connected in virtue of (4.13) and (4.16). So X is connected and, consequently, $\dim X \geqslant -1$.

It remains to show that $tr\text{-}\operatorname{Ind} X = \infty$. For this it suffices to check that

$$X \text{ contains no closed connected compact subset } A \text{ with } \operatorname{Ind} A = 1. \tag{4.18}$$

Assume that there is a closed connected set $A \subset X$ with $\operatorname{Ind} A = 1$. Let $f_n : X \to X_n$ be the limit projections of the inverse sequence S, $n = 1, 2, \ldots$. From Lemma 4.7 and Question 4.16 it follows that f_n are monotonic mapping, i.e.

$$f_n^{-1} x \text{ is connected for each } x \in X_n. \tag{4.19}$$

There is n such that $f_n(A)$ is a non-degenerate continuum. From Lemma 4.7, Question 4.15, and the construction of the inverse sequence S it follows that

$$A = f_n^{-1} f_n(A).$$

Take different points $x_0, x_1 \in f_n(A)$ and put $A_i = f_n^{-1} x_i$, $i = 0, 1$. Then A_0 and A_1 are disjoint closed subsets of A. Since $\operatorname{Ind} A = 1$, there is a partition P in A between A_0 and A_1 such that $\operatorname{Ind} P \leqslant 0$.

All sets $f_k(A)$, $k \geqslant n$, are connected. So Lemma 4.7 implies that all mappings

$$f_k^{k+1} : f_{k+1}(A) \to f_k(A), \quad k \geqslant n,$$

are irreducible. Moreover, they are monotonic. Hence all limit mappings

$$f_k|_A : A \to f_k(A), \quad k \geqslant n,$$

are irreducible and monotonic. Then from [5] it follows that the sets

$$f_k(P), \quad k \geqslant n,$$

are partitions in $f_k(A)$ between $f_k(A_0)$ and $f_k(A_1)$. In particular, $f_{n+1}(P)$ is a partition in $f_{n+1}(A)$ between $f_{n+1}(A_0)$ and $f_{n+1}(A_1)$.

But the mapping $f_n^{n+1}|_{f_{n+1}(A)} : f_{n+1}(A) \to f_n(A)$ satisfies all conditions of Lemma 4.9. Consequently, $f_{n+1}(P)$ contains the inverse image of some point $x \in X_n$.

Applying Lemma 4.7, by induction we prove that

$$f_{n+k}(P) \supset (f_n^{n+k})^{-1} x.$$

Consequently, P contains non-degenerate connected set $f_n^{-1} x$. But this contradicts to the condition $\operatorname{Ind} P \leqslant 0$. Property (4.18) and Theorem 4.4 are proved. □

In connection with conditions (4.1)–(4.8) and Theorem 4.3 the following questions arise.

4.12. Question. Does the assertion of Theorem 4.3 hold in **ZFC**?

4.13. Question. Does there exist a **ZFC**-example of a perfectly normal compact space X with $\dim X = 1$ and $tr\text{-}\operatorname{Ind} X = \infty$?

4.14. Remark. A positive answer to Question 4.13 would imply a positive answer to Question 4.12.

Question 4.13 can be weakened as follows.

4.15. Question. Does there exist a **ZFC**-example of a hereditarily normal compact space X with $\dim X = 1$ and $tr\text{-}\operatorname{Ind} X = \infty$?

This question has the following weakest version.

4.16. Question. Does there exist a **ZFC**-example of a hereditarily normal compact space X with $\dim X < \operatorname{Ind} X$?

References

[1] V.V. Chatyrko, On snakelike and homogeneous compact spaces with noncoinciding dimensions, Uspekhi Mat. Nauk 39 (3) (1984) 247–248.
[2] V.V. Chatyrko, On compact spaces with noncoinciding dimensions, Tr. Mosk. Mat. Obs. 53 (1990) 192–228.
[3] R. Engelking, Theory of Dimensions. Finite and Infinite, Sigma Ser. Pure Math., vol. 10, Heldermaun, Lemgo, 1995.
[4] V.V. Fedorchuk, Mappings that do not reduce dimension, Soviet Math. Dokl. 10 (1969) 314–317.
[5] V.V. Fedorchuk, Compact spaces without intermediate dimensions, Soviet Math. Dokl. 14 (1973) 1808–1811.
[6] V.V. Fedorchuk, On the dimension of hereditarily normal spaces, Proc. London Math. Soc. (3) 36 (1978) 163–175.
[7] V.V. Fedorchuk, Spaces which are weakly infinite-dimensional modulo simplicial complexes, Vestnik Moskov. Univ. Ser. Mat. Mekh. 3 (2009) 33–40.
[8] V.V. Fedorchuk, Finite-dimensional spaces via simplicial complexes and polyhedra, Mat. Vesnik 61 (1) (2009) 25–52.
[9] V.V. Fedorchuk, Inductive dimensions modulo simplicial complexes and *ANR*-compacta, Colloq. Math. 120 (2) (2010) 223–247.
[10] V.V. Fedorchuk, Questions on dimensions modulo simplicial complexes. I. Finite-dimensional spaces, Questions Answers Gen. Topology 28 (1) (2010) 1–25.
[11] V.V. Fedorchuk, Questions on dimensions modulo simplicial complexes. III. Transfinite dimensions, Questions Answers Gen. Topology 28 (2) (2010) 109–132.
[12] K. Kuratowski, Sur les espaces localement connexes et péaniens en dimension n, Fund. Math. 24 (1935) 269–287.
[13] K. Morita, On generalizations of Borsuk's homotopy extension theorem, Fund. Math. 88 (1975) 1–6.
[14] M. Starbird, The Borsuk homotopy extension theorem without binormality condition, Fund. Math. 87 (1975) 207–211.
[15] P. Vopěnka, On dimension of compact spaces, Czechoslovak Math. J. 8 (3) (1958) 319–327.